# Self-stabilizing Iterative Solvers

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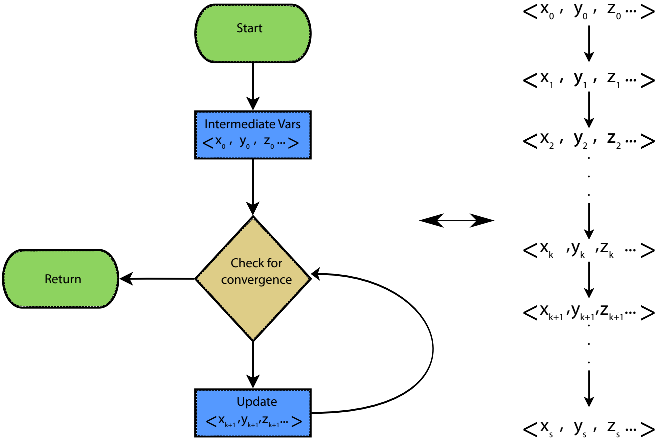
# Self-stabilization

Informally, **self-stabilization** (Dijkstra 1974) is property of system to resume correct behavior no matter what the initial state is given.

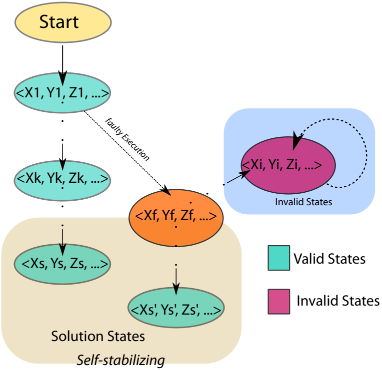
- We describe a self-stabilizing version the conjugate gradients method, which is resilient to transient soft faults.

# Iterative Algorithms



# Self-stabilizing Algorithms

- An algorithm is *self-stabilizing*, if starting from any state (valid or invalid), it comes back to a valid state within finite number of "steps" , otherwise not.




*Self-stabilizing Algorithms*

# Making an Algorithm Self-stabilizing

- Naturally self-stabilizing (e.g., Newton, SOR, Jacobi)
- Restart from a checkpoint
- Restart (such as restarted-GMRES)
- *Our strategy: Correction step*

# Periodic correction step

- Restore sufficient conditions for convergence
- Mathematically "equivalent" to original in a fault-free execution
- Eliminates need for detecting faults
- Executing correction step periodically ensures resuming correct behavior in finite number of steps

# Conjugate Gradient Algorithm

- Solve $Ax = b$ for $x$ for SPD $A$;
- Quadratic optimization problem

$$F(x) = \frac{1}{2}x^T Ax - x^T b$$

- $F(x)$ represents N-dimensional paraboloid
- CG finds the optimum by taking appropriately constructed steps

# Conjugate Gradient (CG) Algorithm

## State variables

- $x_k$ = present estimate
- $p_k$ = search direction
- $r_k = b - Ax_k$ = direction of steepest descent

## Transition function

1. $q_k \leftarrow Ap_k$
2. $\alpha_k \leftarrow \frac{r^T r}{p^T q}$
3. $x_{k+1} \leftarrow x_k + \alpha_k r_k$
4. $r_{k+1} \leftarrow r_k - \alpha_k q_k$
5. $\beta_k \leftarrow \frac{\|r_{k+1}\|^2}{\|r_k\|^2}$
6. $p_{k+1} \leftarrow r_{k+1} + \beta_k p_k$

# Self-stabilizing Conjugate Gradient

- It is a Krylov subspace method, $\{p_k\}$, $\{r_k\}$ spans Krylov subspace $\mathcal{K}(A, r_0, m)$

$$\mathcal{K}(A, r_0, m) = span\{r_0, Ar_0, \ldots A^{m-1} r_0\}$$

- Global orthogonality properties

$$\begin{aligned} p_i^T A p_j &= 0 \quad \text{if } i \neq j; \\ r_i^T r_j &= 0 \quad \text{if } i \neq j; \text{ and} \\ r_i^T p_j &= 0 \quad \text{if } i > j. \end{aligned}$$

- Finite termination in exact arithmetic

# Effects of faults on Conjugate Gradient

- In general, most of Krylov subspace properties are lost
- Multiple potential outcomes due to faults
  1. Error in $r_k \Rightarrow$ Converge to incorrect value
  2. Error in $p_k \Rightarrow$ Diverge, stagnation, slow convergence
- Difficult to detect validity of state

# Self-stabilizing Conjugate Gradient

We identify the following relations that are sufficient to guarentee convergence (a corollary to Zoutendijk condition)

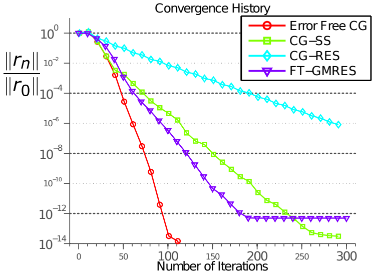
- Residual condition : $r_k = b - Ax_k$
- Optimal step length : $\alpha_k = \frac{r_k^T p_k}{p_k^T A p_k}$
- Correct search direction : $\frac{\left(p_k^T r_k\right)}{\|p_k\| \|r_k\|} > c_1$
- Local orthogonality relation : ${p_{k+1}}^T A p_k = 0$

# Experiments

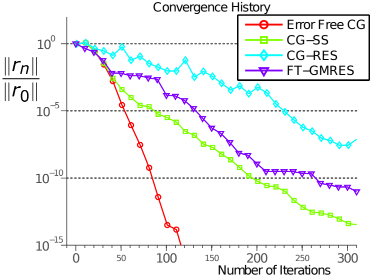
- Assume: *selective reliability mode*, i.e., correction step can be done reliably
- Inject faults in sparse matrix-vector (SpMV) product by flipping bits in matrix entry at a specified rate
  1. Bit flips in mantissa and sign bits - 40 bit flips in every unreliable SpMV
  2. Bit flips can occur any where (including exponent) - 4 bit flips in every unreliable SpMV

# K3D : Quadratic Convergence

- In presence of faults, only linear convergence is observed



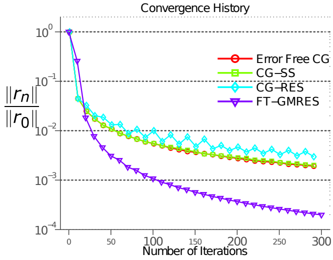

(a) Bounded errors (mantissa and sign bit flips only)



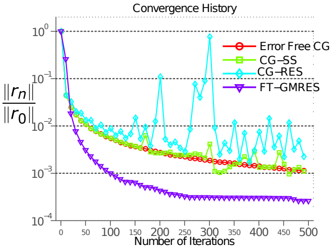

(b) Unbounded errors (including exponent also)

# THERMAL1 : Sub-linear Convergence

- Convergence rate for CG-SS and CG-RES does not change by much; FT-GMRES shows better convergence due to pre-conditioning



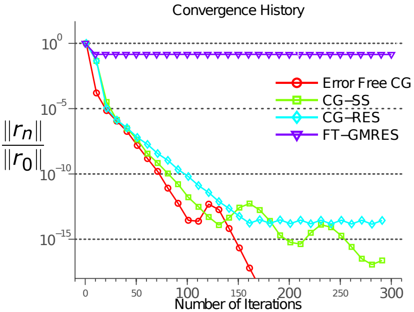

(c) Bounded errors (mantissa and sign bit flips only)



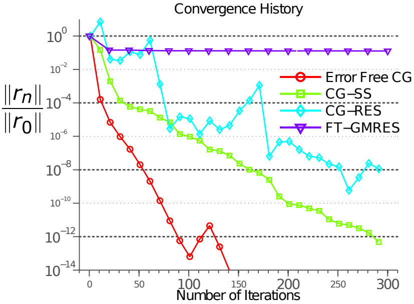

(d) Unbounded errors (including exponent also)

# DIAG : Linear Convergence

- Linear convergence is maintained. However, slight slow-down in convergence is observed



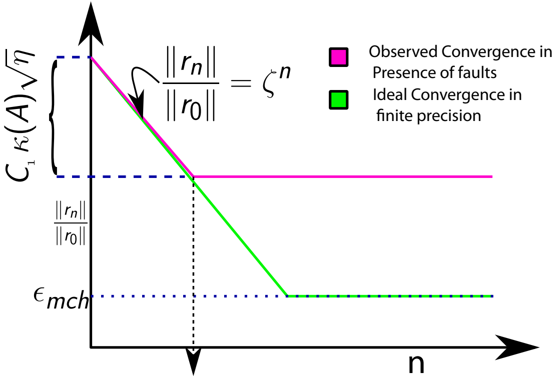

(e) Bounded errors (mantissa and sign bit flips only)




(f) Unbounded errors (including exponent also)

# Analysis

- Can show that if $\kappa(A)$ is the condition number and $\eta$ is rate of bit flips per SpMV, then




$$i_{opt} = log_{\zeta}\left(C_1 \kappa(A)\sqrt{\eta}\right)$$

# Conclusion & Future Work

## Conclusion

- Two mathematically equivalent algorithms can have different resilience. Concept of self-stabilization may give hints
- Self-stabilized CG tolerates high rates of faults
  1. Trade off between reliable and unreliable computation
  2. Reliable computation varies logarithmically with fault rates $\implies$ Scalable

## Future Work

- The balance between reliable and unreliable computation : to optimize for energy / power / time
- Hybridized fault-tolerance techniques